

# On Cox-Kemperman moment inequalities for independent centered random variables

P.S.Ruzankin<sup>1</sup>

## Abstract

In 1983 Cox and Kemperman proved that  $\mathbf{E}f(\xi) + \mathbf{E}f(\eta) \leq \mathbf{E}f(\xi + \eta)$  for all functions  $f$ , such that  $f(0) = 0$  and the second derivative  $f''(y)$  is convex, and all independent centered random variables  $\xi$  and  $\eta$  satisfying certain moment restrictions. We show that the minimal moment restrictions are sufficient for the inequality to be valid, and write out a less restrictive condition on  $f$  for the inequality to hold.

Besides, Cox and Kemperman (1983) found out the optimal constants  $A_\rho$  and  $B_\rho$  for the inequalities  $A_\rho(\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho) \leq \mathbf{E}|\xi + \eta|^\rho \leq B_\rho(\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho)$ , where  $\rho \geq 1$ ,  $\xi$  and  $\eta$  are independent centered random variables. We write out similar sharp inequalities for symmetric random variables.

*Keywords:* Cox-Kemperman inequalities, moment inequalities, centered random variable, symmetric random variable, two-point distribution.

## 1. Introduction and formulation of the results

Cox and Kemperman have proved the following theorem:

**Theorem A** [Cox and Kemperman, 1983].

Let random variables  $\xi$  and  $\eta$  be such that

$$\mathbf{E}(\xi|\eta) = 0, \quad \mathbf{E}(\eta|\xi) = 0 \quad a.s. \quad (1)$$

Then, for each  $\rho \geq 1$ , the following inequalities hold:

$$2^{\rho-2}(\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho) \leq \mathbf{E}|\xi + \eta|^\rho \leq \left( \max_{0 \leq z \leq 1} \psi(\rho, z) \right) (\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho) \quad \text{if } 1 \leq \rho \leq 2, \quad (2)$$

$$\left( \min_{0 \leq z \leq 1} \psi(\rho, z) \right) (\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho) \leq \mathbf{E}|\xi + \eta|^\rho \leq 2^{\rho-2}(\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho) \quad \text{if } 2 \leq \rho \leq 3, \quad (3)$$

$$\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho \leq \mathbf{E}|\xi + \eta|^\rho \leq 2^{\rho-2}(\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho) \quad \text{if } \rho \geq 3 \quad (4)$$

whenever  $\mathbf{E}|\xi|^\rho < \infty$  and  $\mathbf{E}|\eta|^\rho < \infty$ , where

$$\psi(\rho, z) = 2^{\rho-1}(z + z^{\rho-1} + (1-z)^\rho) / ((1+z)(1+z^{\rho-1})).$$

All the estimates in (2) – (4) for  $\mathbf{E}|\xi + \eta|^\rho$  are sharp in the sense that, for each inequality, there exist distributions of  $\xi$  and  $\eta$ , such that  $\xi \neq 0$  and the inequality turns into equality for independent  $\xi$  and  $\eta$  with these distributions.

This theorem does not consider the case  $0 < \rho < 1$  because in the case the sharp inequalities are trivial ones:  $0 \leq \mathbf{E}|\xi + \eta|^\rho \leq \mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho$ .

Besides, Cox and Kemperman (1983) noted that, for i.i.d.  $\xi$  and  $\eta$  having a symmetric two-point distribution,

$$\mathbf{E}|\xi + \eta|^\rho = 2^{\rho-2}(\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho) \quad \text{for all } \rho > 0. \quad (5)$$

It is also known (e.g. see Rosenthal (1972)) that, for symmetric independent random variables  $\xi$  and  $\eta$ ,

$$\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho \leq \mathbf{E}|\xi + \eta|^\rho \quad \text{if } \rho \geq 2 \quad (6)$$

when  $\mathbf{E}|\xi|^\rho < \infty$  and  $\mathbf{E}|\eta|^\rho < \infty$ .

<sup>1</sup>Sobolev Institute of Mathematics, Novosibirsk State University, pr. Ak. Koptuyuga, 4, Novosibirsk, Russia, ruzankin@math.nsc.ru

Besides of estimates for expectations of power functions, there have been obtained certain inequalities for expectations for some other classes of functions.

**Theorem B** [Cox and Kemperman, 1983]. *Let a function  $f$  on the real line be such that  $f(0) \leq 0$  and the second derivative  $f''(y)$  exists for all  $y$  and is convex. Let random variables  $\xi$  and  $\eta$  satisfy condition (1) and be such that*

$$\mathbf{E}|f'(\xi)| < \infty, \quad \mathbf{E}|f'(\eta)| < \infty. \quad (7)$$

*Then*

$$\mathbf{E}f(\xi) + \mathbf{E}f(\eta) \leq \mathbf{E}f(\xi + \eta).$$

Note that by Theorem E below condition (7) can be omitted for independent  $\xi$  and  $\eta$ .

Appendix A below contains a simple proof of this theorem proposed by Borisov. (The proof is valid under certain moment restrictions which can also be omitted for independent  $\xi$  and  $\eta$  by Theorem E.)

Note that the function  $f(y) = |y|^\rho$  satisfies the conditions of this theorem only if  $\rho = 2$  or  $\rho \geq 3$ .

Utev has obtained the following corresponding result:

**Theorem C** [Utev, 1985]. *Let a function  $f$  on the real line have  $f''(y)$  for all  $y$ . Then the following three conditions are equivalent*

- 1).  $f''$  is convex.
- 2). *For all independent symmetric random variables  $\xi$  and  $\eta$  and for all  $y$ , the inequality*

$$\mathbf{E}f(y + \xi) + \mathbf{E}f(y + \eta) \leq f(y) + \mathbf{E}f(y + \xi + \eta)$$

*holds whenever these expectations exist.*

- 3). *For all independent bounded random variables  $\xi$  and  $\eta$ , such that  $\mathbf{E}\xi = \mathbf{E}\eta = 0$ , and for all  $y$ , the inequality*

$$\mathbf{E}f(y + \xi) + \mathbf{E}f(y + \eta) \leq f(y) + \mathbf{E}f(y + \xi + \eta)$$

*holds.*

In fact, Utev formulated and proved this theorem for Hilbert space - valued random variables. However such spaces will not be discussed in the present paper.

Note that the restriction that the random variables  $\xi$  and  $\eta$  in condition 3) be bounded can be omitted by Theorem E below.

Another class of functions is considered in the following statement.

**Theorem D** [Figiel, Hitczenko, Johnson, Schechtman, and Zinn, 1997]. *Let a function  $f$ ,  $f(0) \leq 0$ , be even and such that the function  $y \mapsto f(\sqrt{|y|})$  is convex. Then*

$$\mathbf{E}f(\xi) + \mathbf{E}f(\eta) \leq \mathbf{E}f(\xi + \eta)$$

*for all random variables  $\xi$  and  $\eta$ , such that the conditional distribution of  $\eta$  under the condition  $\xi = x$  is symmetric for all  $x$ .*

Note that the function  $|y|^\rho$  satisfies conditions of this theorem only if  $\rho \geq 2$ .

We will call a function  $f$  on the real line *twice differentiable* if  $f'(y)$  exists for all  $y$ ,  $f''(y)$  exists for almost all (with respect to the Lebesgue measure)  $y$ , and, for all  $a < b$ ,  $f'(b) - f'(a) = \int_a^b f''(y)dy$ .

**Theorem 1.** *Let a function  $f$  be twice differentiable,  $f(0) \leq 0$ , and the function  $f''(t) + f''(-t)$  be nondecreasing for  $t > 0$ .*

*Then, for all independent symmetric random variables  $\xi$  and  $\eta$ ,*

$$\mathbf{E}f(\xi) + \mathbf{E}f(\eta) \leq \mathbf{E}f(\xi + \eta)$$

*whenever the expectations exist.*

A useful corollary of the theorem is the following one.

**Corollary 1.** *For independent symmetric random variables  $\xi$  and  $\eta$ , the following inequalities are valid:*

$$2^{\rho-2}(\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho) \leq \mathbf{E}|\xi + \eta|^\rho \leq \mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho \quad \text{if } 0 < \rho \leq 2, \quad (8)$$

$$\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho \leq \mathbf{E}|\xi + \eta|^\rho \leq 2^{\rho-2}(\mathbf{E}|\xi|^\rho + \mathbf{E}|\eta|^\rho) \quad \text{if } \rho \geq 2 \quad (9)$$

*when  $\mathbf{E}|\xi|^\rho < \infty$  and  $\mathbf{E}|\eta|^\rho < \infty$ . The four estimates for  $\mathbf{E}|\xi + \eta|^\rho$  are sharp in the sense that, for each inequality, there exist distributions of  $\xi$  and  $\eta$ , such that  $\xi \neq 0$  and the inequality turns into equality for these distributions.*

Note that (9) and (8) for  $\rho = 1$  follow directly from Theorem A and relations (5) and (6). Besides, the left inequality in (8) for  $1 \leq \rho \leq 2$  follows directly from Theorem A and relation (5).

Note also that the right inequality in (8) for  $0 < \rho \leq 1$  is trivial because  $|\alpha + \beta|^\rho \leq |\alpha|^\rho + |\beta|^\rho$  for any real numbers  $\alpha$  and  $\beta$ ,  $0 < \rho \leq 1$ .

**Theorem 2.** *Let a function  $f$  be twice differentiable and such that  $f(0) \leq 0$  and*

$$f''(-\alpha) + f''(\beta) \geq f''(-\alpha + \gamma) + f''(\beta - \gamma), \quad (10)$$

*for any  $\alpha > 0$ ,  $\beta > 0$ ,  $0 < \gamma < \alpha + \beta$  such that  $f''$  is defined at the points  $-\alpha$ ,  $\beta$ ,  $-\alpha + \gamma$  and  $\beta - \gamma$ .*

*Then, for all independent centered random variables  $\xi$  and  $\eta$ ,*

$$\mathbf{E}f(\xi) + \mathbf{E}f(\eta) \leq \mathbf{E}f(\xi + \eta)$$

*whenever the expectations exist.*

Note that if  $f''$  is convex then it satisfies the condition (10). But the class of functions subject to condition (10) is wider than the class of functions with convex second derivative.

For instance, if  $f''(y) = \lfloor h(y) \rfloor$ , where  $h(y)$  is nonnegative and convex,  $h(0) = 0$ , then  $f(y)$  satisfies condition (10).  $\lfloor \cdot \rfloor$  denotes integer part of a number.

As another example, we can take  $f''(y) = -y$  if  $y < 1$ ,  $f''(y) = \lfloor y \rfloor - (y - \lfloor y \rfloor)$  if  $y \geq 1$ . Such  $f(y)$  also satisfies (10).

**Remark.** In Theorem 2, if  $\xi + \eta \in [-B, C]$  a.s. then it is sufficient to require that the function  $f$  satisfy condition (10) only for  $\alpha, \beta$  lying in  $(-B, C)$ .

In Theorem 1, if  $\xi + \eta \leq C$  a.s. then it is sufficient to require that  $f''(t) + f''(-t)$  be nondecreasing for  $0 < t < C$ .

The proof of Theorem 1 is based on the fact that any symmetric distribution can be “decomposed” into a mixture of symmetric distributions (e.g. see Figiel, Hitczenko et al. (1997)). As for Theorem 2, any centered distribution can be “decomposed” into a mixture of two-point centered distributions (e.g. see Pinelis (2009) and references therein). So one can prove the corresponding inequalities for two-point distributions only:

**Theorem E.** *Let a function  $g$  of  $m + n$  arguments,  $m \geq 0$  and  $n \geq 0$ , be such that*

$$\mathbf{E}g(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n) \geq 0$$

*for all independent random variables  $\xi_j$  and  $\eta_j$ , where each of the random variables  $\xi_j$  has a centered two-point distribution or equals zero, and each of  $\eta_j$  has a symmetric two-point distribution or equals zero.*

*Then, for all independent random variables  $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n$ , where the random variables  $\xi_j$  are centered and  $\eta_j$  are symmetric, the following inequality is valid:*

$$\mathbf{E}g(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n) \geq 0$$

*whenever the expectation exists.*

## 2. Proofs

### 2.1. Proof of Theorem E

For the sake of convenience we give here the proof of Theorem E, but for the case  $m = 2$ ,  $n = 0$  only. The case of arbitrary  $m$  and  $n$  can be considered analogously.

Denote  $\xi = \xi_1$ ,  $\eta = \xi_2$ . If  $\xi$  and  $\eta$  have centered two-point distributions,  $\xi$  takes values  $-a, b$  and  $\eta$  takes values  $-c, d$  then

$$\mathbf{P}(\xi = -a) = b/(a + b), \quad \mathbf{P}(\xi = b) = a/(a + b), \quad \mathbf{P}(\eta = -c) = d/(c + d), \quad \mathbf{P}(\eta = d) = c/(c + d).$$

Thus we have

$$\mathbf{E}f(\xi, \eta) = \frac{1}{(a + b)(c + d)} (bd f(-a, -c) + bc f(-a, d) + ad f(b, -c) + ac f(b, d)) \geq 0 \quad (11)$$

for all  $a, b, c, d > 0$ .

Now let  $\xi$  and  $\eta$  have arbitrary centered distributions such that  $\mathbf{P}(\xi \neq 0) = \mathbf{P}(\eta \neq 0) = 1$ . Put

$$p = \mathbf{P}(\xi > 0), \quad F_\xi(u) = \mathbf{P}(\xi < u) - (1 - p), \quad G_\xi(u) = \mathbf{P}(-\xi < u) - p.$$

Put also

$$s(y) = \int_0^y F_\xi^{(-1)}(u) du, \quad t(x) = \int_0^x G_\xi^{(-1)}(u) du,$$

where  $F_\xi^{(-1)}(u) := \sup\{x : F_\xi(x) < u\}$  is the quantile transformation of  $F_\xi$ ,

Then  $s(y)$  and  $t(x)$  are (strictly) increasing continuous functions on  $[0, p]$  and  $[0, 1 - p]$ , respectively, and

$$s(p) = \mathbf{E} \max\{0, \xi\} = \mathbf{E} \max\{0, -\xi\} = t(1 - p).$$

Put

$$z(y) = t^{-1}(s(y)).$$

We have  $t(z(y)) = s(y)$ , hence  $dt(z(y)) = ds(y)$  which can be rewritten as

$$G_\xi^{(-1)}(z(y)) dz(y) = F_\xi^{(-1)}(y) dy.$$

For a function  $h$ ,

$$\mathbf{E}h(\xi) = \int_0^p h(F_\xi^{(-1)}(y)) dy + \int_0^{1-p} h(-G_\xi^{(-1)}(x)) dx.$$

Substituting  $x = z(y)$  into the last integral yields

$$\mathbf{E}h(\xi) = \int_0^p \left( h(F_\xi^{(-1)}(y)) + h(-G_\xi^{(-1)}(z(y))) \frac{F_\xi^{(-1)}(y)}{G_\xi^{(-1)}(z(y))} \right) dy. \quad (12)$$

Let us introduce the same notations for  $\eta$ . Put

$$q = \mathbf{P}(\eta > 0), \quad F_\eta(t) = \mathbf{P}(\eta < u) - (1 - q), \quad G_\eta(t) = \mathbf{P}(-\eta < u) - q,$$

and let  $w(v)$  be defined by the relations

$$w(0) = 0, \quad G_\eta^{(-1)}(w(v)) dw(v) = F_\eta^{(-1)}(v) dv.$$

Using the above notations we can write

$$\mathbf{E}g(\xi, \eta) = \int_0^q \int_0^p \psi(y, v) dy dv,$$

where

$$\begin{aligned} \psi(y, v) := & g(F_\xi^{(-1)}(y), F_\eta^{(-1)}(v)) + g(F_\xi^{(-1)}(y), -G_\eta^{(-1)}(w(v))) \frac{F_\eta^{(-1)}(v)}{G_\eta^{(-1)}(w(v))} + \\ & g(-G_\xi^{(-1)}(z(y)), F_\eta^{(-1)}(v)) \frac{F_\xi^{(-1)}(y)}{G_\xi^{(-1)}(z(y))} + g(-G_\xi^{(-1)}(z(y)), -G_\eta^{(-1)}(w(v))) \frac{F_\xi^{(-1)}(y)}{G_\xi^{(-1)}(z(y))} \frac{F_\eta^{(-1)}(v)}{G_\eta^{(-1)}(w(v))}, \end{aligned}$$

and  $\psi(y, v) \geq 0$  by relation (11).

We have proved the statement of the theorem for the case  $\mathbf{P}(\xi \neq 0) = \mathbf{P}(\eta \neq 0) = 1$ . The case  $\mathbf{P}(\xi = 0) > 0$  or  $\mathbf{P}(\eta = 0) > 0$  can be easily dealt with using mixtures of zero and nonzero distributions.

## 2.2. Proof of Theorem 2

Without loss of generality we can assume  $f(0) = 0$ .

By Theorem E it is sufficient to prove the statement of Theorem 2 for all  $\xi$  and  $\eta$  with centered two-point distributions.

Take  $\xi \in \{-a, b\}$ ,  $\eta \in \{-c, d\}$ , where  $a, b, c, d > 0$ . We have

$$\mathbf{E}f(\xi + \eta) - \mathbf{E}f(\xi) - \mathbf{E}f(\eta) = \frac{1}{(a+b)(c+d)} \phi(a, b, c, d),$$

where

$$\begin{aligned} \phi(r, s, t, u) &= su \left( f(-r-t) - f(-r) - f(-t) \right) \\ &+ st \left( f(-r+u) - f(-r) - f(u) \right) \\ &+ ru \left( f(s-t) - f(s) - f(-t) \right) \\ &+ rt \left( f(s+u) - f(s) - f(u) \right). \end{aligned}$$

Note that  $\phi(r, s, t, u) = 0$  if  $r = 0$  or  $s = 0$  or  $t = 0$  or  $u = 0$ . Moreover,

$$\frac{\partial^4}{\partial r \partial s \partial t \partial u} \phi(r, s, t, u) = f''(-r-t) + f''(s+u) - f''(-r+u) - f''(s-t) \geq 0$$

for positive  $r, s, t, u$  by condition (10).

Therefore

$$\begin{aligned} 0 &\leq \int_0^d \int_0^c \int_0^b \int_0^a \frac{\partial^4}{\partial r \partial s \partial t \partial u} \phi(r, s, t, u) dr ds dt du = \\ &\sum_{r \in \{0, a\}, s \in \{0, b\}, t \in \{0, c\}, u \in \{0, d\}} (-1)^{\text{sgnr} + \text{sgns} + \text{sgnt} + \text{sgnu}} \phi(r, s, t, u) = \phi(a, b, c, d), \end{aligned}$$

where  $\text{sgnr} = 0$  if  $r = 0$  and  $\text{sgnr} = 1$  if  $r > 0$ . Thus

$$\phi(a, b, c, d) \geq 0,$$

and hence the theorem is proved.

### 2.3. Proof of Theorem 1

The proof is analogous to that of Theorem 2.

Without loss of generality we can assume  $f(0) = 0$ .

By Theorem E it is sufficient to prove the statement of Theorem 1 for all  $\xi$  and  $\eta$  with symmetric two-point distributions.

Take  $\xi \in \{-a, a\}$ ,  $\eta \in \{-b, b\}$ , where  $a, b > 0$ . We have

$$\mathbf{E}f(\xi + \eta) - \mathbf{E}f(\xi) - \mathbf{E}f(\eta) = \frac{1}{4}\phi(a, b),$$

where

$$\begin{aligned} \phi(r, s) &= (f(-r-s) - f(-r) - f(-s)) + (f(-r+s) - f(-r) - f(s)) \\ &\quad + (f(r-t) - f(r) - f(-s)) + (f(r+s) - f(r) - f(s)). \end{aligned}$$

Further,

$$\frac{\partial^2}{\partial r \partial s} \phi(r, s) = f''(-r-s) + f''(r+s) - f''(-r+s) - f''(r-s) \geq 0$$

for positive  $r, s$  because  $f''(t) + f''(-t)$  is nondecreasing for positive  $t$ .

Thus

$$0 \leq \int_0^b \int_0^a \frac{\partial^2}{\partial r \partial s} \phi(r, s) dr ds = \phi(a, b) - \phi(a, 0) - \phi(0, b) + \phi(0, 0) = \phi(a, b).$$

Therefore  $\phi(a, b) \geq 0$ , and hence the theorem is proved.

### 2.4. Proof of Corollary 1

In the case  $1 < \rho < 2$  the function  $f(y) = -|y|^\rho$  satisfies the conditions of Theorem 1. Hence, (8) is valid for  $1 < \rho < 2$ .

It remains to show that the left inequality in (8) holds for  $0 < \rho < 1$ . By Theorem E it suffices to show that, for any  $a$  and  $b$ ,  $0 < a < b$ ,

$$\phi(a, b) := (a+b)^\rho + (b-a)^\rho - 2^{\rho-1}a^\rho - 2^{\rho-1}b^\rho \geq 0.$$

We have  $\phi(a, b) = a^\rho h(z)$ , where  $z = b/a$ ,

$$h(z) = (1+z)^\rho + (z-1)^\rho - 2^{\rho-1} - 2^{\rho-1}z^\rho,$$

Thus  $h(z) \geq 0$  for  $z \geq 1$  because  $h(1) = 0$ ,  $h'(z) > 0$  for  $z \geq 1$ .

Hence, (8) is valid for  $0 < \rho < 1$ .

The corollary is proved.

## Appendix A. A simple proof of Theorem B

I.S. Borisov in an oral conversation has proposed the following proof of Theorem B. We have

$$f(\xi + \eta) = f(\xi) + f'(\xi)\eta + \eta^2 \int_0^1 (1-z)f''(\xi + z\eta)dz.$$

Now let us consider the expectation of the last integral. By convexity of  $f''$ ,

$$\mathbf{E} \left( \eta^2 \int_0^1 (1-z)f''(\xi + z\eta)dz \mid \eta \right) \geq \eta^2 \int_0^1 (1-z)f''(z\eta)dz = f(\eta) - f(0) - f'(0)\eta.$$

Thus

$$\mathbf{E}f(\xi + \eta) \geq \mathbf{E}f(\xi) + \mathbf{E}f(\eta) + \mathbf{E}f'(\xi)\eta - \mathbf{E}f'(0)\eta.$$

The restriction of this proof is that all the needed moments, such as  $\mathbf{E}\eta^2 f''(\xi + z\eta)$  for  $0 < z < 1$ , must exist. For, roughly speaking, “regular” functions  $f(y)$  growing not faster than  $e^{c|y|}$ ,  $c = \text{const}$ , the existence of the moments follows from the monotonicity of the functions  $f''(y)$ ,  $f'(y)$ ,  $f(y)$  for sufficiently large  $y$  and for sufficiently large  $-y$ .

As was noted above, for independent  $\xi$  and  $\eta$ , these moment restrictions can be omitted by virtue of Theorem E.

## References

- [1] Cox, D.C. and Kemperman, J.H.B. (1983) Sharp bounds on the absolute moments of a sum of two i.i.d. random variables // Ann. Probab., Vol.11 No. 3, pp. 765–771
- [2] Figiel, T., Hitczenko, P., Johnson, W. B., Schechtman, G., and Zinn J. (1997) Extremal properties of rademacher functions with applications to the Khintchine and Rosenthal inequalities // Transactions of the American Mathematical Society, Vol. 349 No. 3, pp. 997–1027.
- [3] Pinelis, I. (2009) Optimal two-value zero-mean disintegration of zero-mean random variables // Electron. J. Probab. 14, pp. 663–727.
- [4] Rosenthal, H.P. (1972) On the span in  $L^p$  of sequences of independent random variables. In: Proceedings of the sixth Berkeley symposium on mathematical statistics and probability. Vol. 2, Berkeley, 1972, pp. 149–167.
- [5] Utev, S.A. (1985) Extremal problems in moment inequalities. In: Limit Theorems in Probability Theory, Trudy Inst. Math., Novosibirsk, 1985, pp. 56–75 (in Russian).